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Combining historical, foundational, and developmental insights to build children's first steps in mathematics

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How can we design mathematical instructional activities that reveal children's early mathematical competence in an analogous way as school activities that exploit children's mother language competence? We put forward a list of mathematical conceptions young children may have been taught previously and some subsequent actions developed to observe them and guide their first steps in mathematics in an instructional context. The list has been developed on the basis of insights from modern axiomatic presentation of arithmetic and geometry contrasted with historical results and epistemology of mathematics. We discuss the application of the list in a singular context, a group of eight 3 to 8 year-old Spanish children with Trisomy 21, to show the suitability of this tool for revealing early mathematical competence.

Keywords: Experiential learning, early years education, special needs, mathematical skills.

The central issue: Enhancing young children's competence in mathematics

The starting point of our research was a situation of stagnation in two different and singular educational contexts: the encounter with mathematics of Italian primary first graders and the special needs of children with Trisomy 21(or Down syndrome).

Actual current praxis in Italian first grade classrooms[1] and the available research and teaching materials regarding mathematics for children with Trisomy 21 show a remarkable similarity in two aspects. Firstly, from the point of view of contents, there is a focus on numeracy and especially on teaching and learning of numerals and written arithmetic and a clear exclusion of geometry. Secondly, from a wider cultural point of view, we witness a marked lack of confidence in the relationship between mathematics and children's feelings and mind. (Millán Gasca & Gil Clemente, 2016). Both aspects are interrelated. As mathematics is socially viewed as a key component of our modernity, school is forced to cope with the difficulties of its teaching and learning. Consequently, teachers must concentrate on the traditional hardcore, that is, numeracy and practical tasks,

[1] A quite homogeneous didactical praxis was identified thanks to the relationship of the Roma Tre University Department of Education with schools (private, state, urban and rural, and with students from different social backgrounds) in the Lazio area in 2006-2014. Although compulsory mathematics teaching or initiation for 3 to 5 years old children is not laid down, in some Italian preschools, including the wide number of state Montessori schools, mathematics is a part of the curriculum and goals. A sharp contrast is experienced between the good results obtained in these latter schools, where pupils usually show a deep interest and curiosity towards number and geometry, and a general situation of difficulty, anxiety and fear of primary school first graders starting in the first months of first grade where schoolwork is focused on exercises about the writing of the numerals from 1 to 9 and moving on to addition and number symbols with two digits from 11 to 20 after 3-4 months.

narrowing the goals of the learning to obtain instructional success. Centrality of arithmetic and abandonment of formative goals are especially noticeable when teaching children with Trisomy 21[2].

Our working hypothesis was that early mathematical competency, in the same way as linguistic competency, could be enhanced and analyzed in terms of naive conceptions. For this aim, we developed a list of items (including concepts and observable actions) to be used in the design of focused mathematical instructional activities, able to bring out young children's mathematical competency (avoiding initiating children in mathematics through written arithmetic), in much the same way as school activities exploit children's mother language competency (avoiding initiating children in linguistic expression through grammar). After testing the suitability of the list with a group of twelve 4 year-old children schooled in Lazio (Italy) (Colella, 2014), we faced the challenge of using it to enhance the mathematical competence of a group of children with Trisomy 21 in Spain.

Naive mathematical conceptions

The fact that *there is a lot of mathematical life* "*before school*" or *"before being taught"* has been pointed out by authors such as Martin Hughes (1986), Margaret Donaldson (1978) and Liliana Tolchinsky (2003). Usage-based theory of toddlers´ language acquisition (Tomasello, 2003) offers a description of the key situations leading to the first holophrases in a joint adult-child attentional frame. It helps us to understand the precocity of children and their interest and enthusiasm regarding numbers as well as geometry.

The empirical examples considered by Hughes among children in Edinburgh's Department of Psychology nursery, as well as by Karen Fuson (1988) regarding her two daughters, recorded in the mother's diary she started at 1 year and 8 months, were observations of what are considered as *naive arithmetical* conceptions, that is, conceptions that have been observed in children independently of schooling and instructional design. Fuson and Hughes go beyond Piaget and his collaborators' work on the roots of children's understanding of arithmetical ideas because they avoided concentrating solely on the search for a developmental path and on the isolation of spontaneous cognitive development. Instead, they tried to come closer to children's thoughts and feelings through close interviews, observations and task experiments, considering that elementary teaching *enters the scene* in the wider context of children's human experience and development.

We have extended this available research in arithmetic to encompass geometry, following René Thom's views (Israel, Millán Gasca, 2012; Millán Gasca 2016), as shown in later pages.

Insights from historical and mathematical perspectives on primitive objects and relationships

In order to identify geometrical items we drew inspiration from Federigo Enriques (1924-27). He focused on the instructional meaning of the identification of the "primordial", primitive, undefined concepts of the modern axiomatic description of arithmetic and geometry considered in their historical context (Israel & Millán Gasca, 2012).

[2] Elisabetta Monari (2002) pointed out that the "old tree of mathematics" (the traditional view of school mathematics as introduction to written arithmetic) should be replaced by a "new tree", where different cognitive potential could appear.

It can be noted that the pivotal role of the acquisition of sequence of number-words in arithmetic is strikingly coherent with the modern Peano's (1899) axiomatic description of natural numbers. Information on undefined primitive objects (*number* and *one*) and their relationship (*successor*) is contained in axioms, such as, "*one is not the successor of any number*" or "*if two numbers have the same successor they are the same number*" that evoke counting. In addition, the first recursive definitions of addition and multiplication and the definition of "greater than" start from this information. Fuson's description of early arithmetical conceptions includes all of these mathematically critical ideas regarding an ordinal view of natural numbers defined in a child's way, together with the cardinal views and measures uses to which children are exposed nowadays (these views were also central in the ancient origins of number and of extensions of the concept of natural number). This attention to mathematically well-identified different number situations in children's experience is crucial to draw indications for instructional activities.

Following the same path, we paid special attention to the role of Hilbertian (1902) undefined concepts (point, straight line, plane), relationships (congruence, lie in, lie between) and first definitions of objects and relations (angle, segment, circle, triangle, polygon, greater than...) deduced from the axioms, in the building of naïve geometrical conceptions in young children´s minds. For example, in relation to the concept of line (paradigm of the continuum) and straight line, possible naïve conceptions regard line as a path; line as a stroke on a sheet of paper with a pencil; lining up, or walking along the minimum distance between two positions. In relation with the concept of point we can observe whether they know to stand at a fixed point, or if they are able to draw them. Talking about change of direction can be a way to introduce the concept of angle.

Inspired by these primitive concepts we considered performance regarding "solving simple geometric problems", such as drawing a circle freehand; cutting a circle; comparing circles (cutting and overlapping); drawing a straight line connecting two points; joining several numbered points freehand with a straight line; drawing a non-straight line; answering questions related to which object is longer, bigger or thicker...

Naïve arithmetical and geometrical conceptions list

The list we have developed, based upon primitive arithmetical and geometrical concepts and the previous developmental insights, includes some possible naïve mathematical conceptions that children can have. Following Fuson's point of view (1988), we looked for a web of conceptions, including connections between geometry and numbers and some obvious relationships, and not for a systematic building of a theory. Naive conceptions also include the ideas on symbols belonging to the oral dimension of language (not centered at all on the decimal positional numeration system.).

As for mother language, naive mathematical conceptions include competence together with errors and misunderstanding in a dynamic setting, where exposure to new experiences or situations helps the child to correct by him/herself previous ideas or accept and include corrections from peers or adults.

From the list, we have also developed a guide for observation in action. This guide consists of several activities intended either to observe this possible mathematical competency (activities simulating a non-instructional "informal" context, such as domestic or playground experiences) or as "opportunities to learn" (becoming proper instructional, teaching activities). These activities should be embedded in children's overall living experience and should have a human sense for them (Donaldson, 1978). Furthermore, they should at the same time, help to actually generate learning, that is, to guide first steps in mathematics. Of course the border between observation and generation of learning is not sharp. For instance, during the time expended in exploring a question, many children may learn something or reinforced their knowledge (taking into account besides that not every child has the same previous informal opportunities to build naïve conceptions).

Relating to arithmetic, activities such as *bringing enough pencils for everybody, telling the cook how many people to prepare lunch* may bring out naïve conceptions, skills children already possess such as *knowing some number words, knowing some part of the number sequence, counting things*. These conceptions are connected with the primitive arithmetical concepts, *idea of number one,* or *successive number*. However, children can also bring into play these conceptions to actually be able to use their knowledge to bring enough or the exact number of pencils or to try to give an answer to the cook or even to answer correctly.

In relation with geometry other activities like *walking down a road*, *holding a thread between two children*, *folding a sheet of paper neatly* are suitable for bringing to light naïve conceptions of *path* and *line,* which are closely related to the primitive concept of *straight line.* In the same way, children can use these conceptions to learn, as instance, how to distinguish a straight line from a curved one.

First steps in mathematics for children with Trisomy 21.

When faced with the mathematical instruction of young children with Trisomy 21, we had to take into account the adverse general context of confusion about goals and contents mentioned at the beginning. In this context, children with Trisomy 21 appear to be in a clear disadvantage due to their well-known difficulties with arithmetic, lack of effective proposals for teaching and misunderstanding of the role of the discipline in their personal development.

There is also a problem in assessing the actual mathematical knowledge of children with Trisomy 21 (Faraguer, 2014) attributed to their scarce skills in oral and written language and their avoiding behaviour when put in stress situations (Wishart, 1993). This has lead to evaluations based upon interviews with parents or professionals (Faraguer, 2014) and consisting of solving decontextualized tasks (Zimpel, 2016). Such evaluations use to show a poor performance in mathematics by people with Trisomy 21.

From the success obtained using the list of naïve conceptions with a group of 4 year-old Italian children with no previous exposure to mathematics (Colella, 2014), this list appeared to be a suitable tool to make a proper assessment of the previous mathematical ideas of the children with Trisomy 21. We could also use this assessment as a basis for the building of an accurate teaching programme, that focus on formative values of mathematics without giving up to placing high expectations on the children.

Methodology

The experience[3] consisted of a twenty-hour workshop over ten months with a group of eight children between 3 and 8 years old (three aged 3, two aged 5, two aged 6, and one aged 8) without previous selection[4]. The workshop was conducted by a team of four volunteer special education teachers and devoted the first three months to an exhaustive exploration of their naïve arithmetic and geometrical conceptions.

It was a *study case*[5] framed in what it is known as *research for practice* (Faraguer, 2014). Throughout the sessions we made an experiential observation, which allowed us to write a narration of the living experience (Van Manen, 2013) and prepare a final description of the naïve conceptions of each child in relation to the items we have observed.

Development of the workshop

Firstly, we have to adapt the original list to make it suitable to the group of children. Table 1 shows the conceptions definitively explored.

Arithmetical conceptions	Geometrical conceptions
Numbers (any ideas) Counting (transitive and intransitive) Cardinality Subitizing Zero Spontaneous symbolic representation of quantity Resolution of simple arithmetical problems	Idea of point Line and idea of continuous Idea of straight and of non-straight Ideas of angle Ideas of round and circle Ideas of triangles and quadrilaterals Ideas of sphere and other regular solid figures Resolution of simple geometrical problems
Measure of time Distance Use of cardinal numbers to measure a distance (steps)	

Table 1: Some arithmetical and geometrical naïve conceptions

Secondly, throughout the three two-hour sessions devoted to the exploration of their naïve mathematical conceptions, we faced the challenge of designing activities also adapted to features of children with Trisomy 21 (for example most of them did not speak, so we could not use dialogue to build mathematical knowledge). We practiced oral sequence when counting balls to decorate a Christmas tree or when counting time playing hide-and-seek. We worked with the concept of

[3] Carried out in the context of the PhD thesis of the second writer devoted to the exploration of geometry with children with Trisomy 21, following a careful consideration of geometry in Édouard Séguin's approach. (Gil Clemente, 2016)

[4] Families who decided to participate in the research, were members of a local association in Zaragoza (Spain) and have more confident outlook than those of older children with a disappointing experience of primary school.

[5] It is a common methodology with children with Trisomy 21, because as Monari (2002) pointed out, these studies open the path to more general ones that usually confirm results obtained in singular cases.

straight line folding a letter to the Wise Men. We walked along paths to discover new worlds or join points to discover secret drawings. We also compared the length of swords before fighting as a way to compare magnitudes. We understood geometrical concepts through mimesis when training for having an adventure and we discovered surprising similarities among different familiar objects (balls, fruits, caps, towers, boxes, tins or tubes…).

We must highlight the importance of applying these activities in a happy play context in which we could witness their individual and group processes of learning without interference or pressure.

Results

In spite of the limitations inherent with an experience carried out in a formal context and not in their real life[6] we obtained some useful conclusions to guide our later research.

Most of the children, especially the youngest, had very limited initial arithmetical conceptions. Only three of them were able to count to nine and the rest hardly know the numbers “one, two, three”. They could only subitize one or two objects, except the eldest child who reached six objects. They made a lot of mistakes reciting oral sequences (they counted objects more than once or forgot objects when counting) and counting objects or drawings (although they counted objects better than drawings). Only the three eldest children had well established some conception of cardinality and these children were able to solve some very simple arithmetic problems (such as “give me n” or answering to the question “n plus m” with low numbers and only by counting). Surprisingly, hardly any had difficulties in understanding zero in several ways, consistently with the research made by Zimpel (2016): most recognized the cipher, some knew that it was the number before one in the number sequence and some said the word “zero”.

However, their initial geometrical conceptions were much better. Through the use of their bodies, movement and mimesis they showed their understanding of point as a fixed position (standing on it without moving), of line as a path (making an effort to go along it without bending) and of straight line as the minimum distance between two points (they all walked straightly when asked to go from one teacher to another). The eldest ones were also able to distinguish none-straight lines and named them as “curves”. They all had an idea of a circle as a round (they knew how to sit in a circle or how to turn on themselves). However, they had scarce ideas of polygons (they showed more difficulties in recognizing triangles than in recognizing quadrilaterals). Surprisingly, they had a special ability to discover the similarities among every day solids. Acquisition of skills related with drawing differed substantially from one child to another due to the delay in motor development common in Trisomy 21.

Their greatest difficulties in geometrical conceptions had to do with every aspect related to measure (counting steps, for instance, was almost impossible for all of them, even for the eldest one) due to the strong relation between measure and numbers. They also showed a poor performance in understanding the relationship “to be between two objects or two persons”, basic for the acquisition of the concept of segment.

[6] We observed only one of the children, the eldest, in his everyday life. From this observation we made a diary that was very useful for extending our research (Gil Clemente, 2016).

The most remarkable conclusion was the enthusiasm and good disposition showed by all the children when facing mathematical tasks: only one child did not engage in the activities proposed; the other children enjoyed the activities and concentrated on them; many families told us their children were looking forward to coming back and doing “mathematics”. This widely confirmed our initial thesis about the natural relationship between mathematics and childhood, even for those with disabilities.

The results of our observation show a path to the possibility of seizing the power of geometry for developing some abstract thinking processes in children with Trisomy 21. This is consistent with the role attributed to geometry by Séguin (1846,1866) for awakening ideas in disabled children’s minds and with recent research regarding the strength of abstraction in Trisomy 21 (Zimpel, 2016).

Final remarks

The experiences carried out with the two groups of children in Italy and in Spain, indicate that this approach to the encounter with mathematics actually stimulates knowledge building on a solid basis by avoiding the non-involvement of children in school mathematics and is therefore a promising path for future research. It runs in contrast with normal standardized numerical school exercises, by proposing items connected to the development of a relationship of *intimacy* with abstract mathematical objects such as points, segments or numbers (Thom, 1971) which should lay the basis for further introduction to symbolic thought.

Introducing geometry in children’s education as a result of the confidence in the relationship between mathematics and childhood[7] helps children to develop this abstract thinking. We have confirmed this idea with the development and application of subsequent teaching sequences based mainly on geometrical concepts after the exploration of the naïve conceptions described (Colella, 2014; Gil Clemente, 2016).

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